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Translated by D.E.B.

PMM U.S.S.R.,Vol.51,No.4,pp.426-430,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON THE MOTION OF CHAPLYGIN'S SLEDGE*

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The plane motion of Chaplygin's sledge is studied. In his original studies of this non-holonomic system, Chaplygin /1/ assume that the support plane is horizontal, and used a reduction factor to reduce the problem to the study of a Hamiltonian system with two degrees of freedom and one cyclical coordinate (i.e., a completely Liouville integrable system). A smooth reversible replacement of the phase variables is used below for the reduction. The motion is studied in detail by the methods of Hamiltonian mechanics, and the motion on an inclined plane is studied by the averaging method. The problem was earlier studied in /2-4/ for certain constraints on the position of the sledge centre of gravity. Chaplygin's equations of motion on an inclined plane were integrated in /3/ on the assumption that the centre of gravity lies on a line through the blade and perpendicular to the blade.

1. We consider the motion of a rigid body supported on a smooth inclined plane by a blade and two smooth roots (a "balanced" Chaplygin sledge), in a homogeneous field of gravity with acceleration g. The oriented space of this system is three-dimensional and can be written as the layer between two parallel planes R^2, opposite points of which are identified /3/, i.e., $M_0 = R^2 \times S^1$.

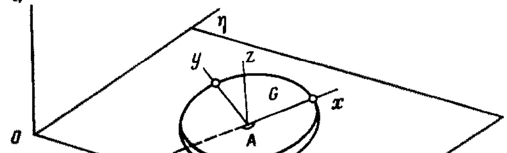

Fig.1

*Prikl.Matem.Mekhan.,51,4,546-551,1987

As the coordinates in M_0 we take the coordinates ξ, η of the blade A in the fixed coordinate system $O\xi\eta\zeta$, the plane $O\xi\eta$ of which is the same as the support plane, the $O\xi$ axis being directed along the line of steepest descent, while $\varphi \bmod 2\pi$ is the angle of rotation of the blade about the $Az \parallel O\zeta$ axis. We also introduce the coordinate system $Axyz$, rigidly connected with the sledge, in such a way that the Ax axis is directed along the blade, and the Ay axis is parallel to the support plane, and the $Axyz$ and $O\xi\eta\zeta$ coordinate systems have the same orientation. We will use the following notation in future: α, β, δ are the coordinates of the sledge centre of gravity G in the connected coordinate system, m is the sledge mass, k is the radius of inertia of the sledge with respect to the $Gz' \parallel Az$ axis, $r = \sqrt{\alpha^2 + k^2}$ is the sledge radius of inertia with respect to the $Bz'' \parallel Az$ axis (B is the projection of the point G onto the Ay axis), and θ is the angle of the support plane to the horizontal (Fig.1). We assume throughout (unless otherwise stated) that $\alpha \neq 0$. The Lagrange function of the system (a smooth function on TM_0) is a tangent fibering of M_0. In the chosen coordinates it has the form /1/

$$L_0 = {}^1/_2 m \{[\xi^{\cdot} - (\alpha \sin \varphi + \beta \cos \varphi)\varphi^{\cdot}]^2 + [\eta^{\cdot} + (\alpha \cos \varphi - \beta \sin \varphi)\varphi^{\cdot}]^2 + k^2\varphi^{\cdot 2}\} + mg \sin \theta\, (\xi + \alpha \cos \varphi - \beta \sin \varphi) \tag{1.1}$$

We impose a non-integrable coupling on the system, i.e., the absolute velocity of blade A is directed along the Ax axis (along the blade), i.e.,

$$\xi^{\cdot} \sin \varphi - \eta^{\cdot} \cos \varphi = 0 \tag{1.2}$$

It is clear from (1.1) and (1.2) that the system is a non-holonomic Chaplygin system and its equations of motion can be integrated in the form of Chaplygin equations independently of the coupling Eq.(1.2). Hence the equations of the system can be conveniently regarded as a mapping $\Delta: R \to M = R^1 \times S^1 \subset M_0$, which satisfies in local coordinates in M the Chaplygin equations, in which the Lagrange function L takes account of the non-integrable coupling. Unfortunately, in ξ and φ coordinates, L has a singularity at $\cos \varphi = 0$.

Following Chaplygin /1/, it is better to start by introducing the quasicoordinate q by

$$\xi^{\cdot} = q^{\cdot} \cos \varphi, \quad \eta^{\cdot} = q^{\cdot} \sin \varphi \tag{1.3}$$

and then the quasicoordinate $\varkappa = q - \beta\varphi$. Notice that

$$\xi^{\cdot} = (\varkappa^{\cdot} + \beta\varphi^{\cdot}) \cos \varphi \tag{1.4}$$

and $q^{\cdot}$ and $\varkappa^{\cdot}$ are respectively the projections onto the Ax axis of the blade A absolute velocity and the centre of gravity G. Obviously, q or $\varkappa$ along with φ no longer define the body position uniquely, but since L depends linearly on ξ, the equations of motion in the quasicoordinates $\varkappa$, φ /3/ will no longer contain ξ explicitly and can be considered independently of (1.3).

The Lagrange function L, which takes account of the coupling (1.3), and the corresponding Hamiltonian, have the form

$$L = {}^1/_2 m (\varkappa^{\cdot 2} + r^2\varphi^{\cdot 2}) + mg \sin \theta\, (\xi + \alpha \cos \varphi - \beta \sin \varphi) \tag{1.5}$$

$$H = (2m)^{-1} (p_\varkappa^{2} + r^{-2}p_\varphi^2) - mg \sin \theta\, (\xi + \alpha \cos \varphi - \beta \sin \varphi) \tag{1.6}$$

The equations of motion in canonical form

$$\begin{aligned}
&\varkappa^{\cdot} = \frac{\partial H}{\partial p_\varkappa} = \frac{p_\varkappa}{m}, \quad p_\varkappa^{\cdot} = -\frac{\partial H}{\partial \varkappa} + \Gamma_\varkappa \quad \left(\Gamma_\varkappa = \frac{\alpha p_\varphi^2}{mr^4}\right)\\
&\varphi^{\cdot} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2}, \quad p_\varphi^{\cdot} = -\frac{\partial H}{\partial \varphi} + \Gamma_\varphi \quad \left(\Gamma_\varphi = -\frac{\alpha p_\varphi p_\varkappa}{mr^2}\right)\\
&\left(\frac{\partial H}{\partial \varkappa} = \frac{\partial H}{\partial \xi}\cos\varphi = -mg\sin\theta\cos\varphi,\ \frac{\partial H}{\partial \varphi} = \alpha mg\sin\theta\sin\varphi\right)
\end{aligned} \tag{1.7}$$

which describe the motion of our non-holonomic system, define the dynamic system in an invariant measure, specified by the density $\mu = e^{\gamma\varkappa}$ ($\gamma \equiv \alpha r^{-2}$). This invariant measure is infinite, The first two integrals are not sufficient for square summability. It seems that in general they are not known. The function H (the total energy of the system) is the first integral of system (1.7) if it is supplemented by relation (1.4).

We know that a system with density $\mu > 0$ can be reduced to a system of density $\mu = 1$ by the change of variable $d\tau = \mu(\varkappa)dt$. It is interesting that, when $\theta = 0$ this change of time along with the linear transformation of momenta $\mathbf{p}' = \mu(\varkappa)\mathbf{p}$ reduces system (1.7) to the form of the usual Hamiltonian equations /1/.

2. Now let $\theta = 0$. Then, Eqs.(1.7) have two first integrals H and $I \equiv p_\varphi e^{\gamma\varkappa}$ and are

square summable. We introduce the quasicoordinate π_2 by the relation /6/

$$\pi_2^{\cdot} = \varphi^{\cdot} e^{-\gamma\varkappa} \tag{2.1}$$

The Lagrange function (1.5) is then $(\pi_1 \equiv \varkappa)$

$$L = {}^1/_2 m\,(\pi_1^{\cdot 2} + r^2 e^{2\gamma\pi_1}\pi_2^{\cdot 2}) \tag{2.2}$$

and calculations show that the equations of motion in quasicoordinates π_i are /3/

$$\frac{d}{dt}\frac{\partial L}{\partial \pi_i^{\cdot}} - \frac{\partial L}{\partial \pi_i} = 0 \quad (i = 1, 2) \tag{2.3}$$

We can also write Eqs.(2.3) in the canonical form

$$p_i^{\cdot} = -\partial H/\partial \pi_i,\ \pi_i^{\cdot} = \partial H/\partial p_i\ (p_i = \partial L/\partial \pi_i^{\cdot}) \tag{2.4}$$

$$H = (2m)^{-1}\,(p_1^2 + r^{-2}e^{-2\gamma\pi_1}p_2^2) \tag{2.5}$$

The formal procedure of introducing quasicoordinates can be interpreted as a smooth (with $p_\varphi \neq 0$) reversible change of phase variables $(\varkappa, \varphi, p_\varkappa, p_\varphi) \to (\pi_1, \pi_2, p_1, p_2)$, which reduces Eqs. (1.7) to the form of ordinary Hamiltonian equations. It has the form

$$\pi_1 = \varkappa, \quad \pi_2 = \varphi + \gamma^{-1}\,[p_\varkappa p_\varphi^{-1} e^{-\gamma\varkappa} + r^{-1}\operatorname{sign}(p_\varkappa p_\varphi)\arcsin(1 + p_\varkappa^2 p_\varphi^{-2} r^2)^{-1/2}], \quad p_1 = p_\varkappa, \quad p_2 = p_\varphi e^{\gamma\varkappa} \tag{2.6}$$

On the right-hand side of the expression for π_2 in (2.6) we have omitted the term $-(2\gamma r)^{-1}\pi\ \operatorname{sign} p_\varkappa$ (which is necessary to ensure smoothness with respect to $p_\varkappa$), since it has no effect on our future arguments.

With $p_\varphi = 0$ $(I = 0)$ the required change is identical, since here again (1.7) have the form of Hamiltonian equations (as is also the case, incidentally, when $\gamma = 0$). Notice that π_2, like φ, is an angular coordinate, i.e., $\pi_2 \bmod 2\pi$.

Relations (2.6) specify a mapping of the phase space T^*M (T^*M is a cotangent fibering of M) into itself. The transformed Hamiltonian (2.5) is a smooth function in T^*M, while the corresponding Eqs.(2.4) describe the motion of our non-holonomic system. Thus we can choose the map (π_1, π_2, p_1, p_2) in the phase space T^*M in such a way that the phase flow trajectories (1.7) are mapped by the integral curves of the usual canonical Eqs.(2.4).

The Hamiltonian system (2.4) has two independent integrals in involution: $H = h = \text{const}$ and $p_2 = c = \text{const}$ and is therefore completely integrable. It can be reduced to a system with one degree of freedom. The reduced phase space is R^2, while the reduced Hamiltonian is obtained from (2.5) by replacing p_2 by c. The domain of possible motions is not empty if $h \geqslant 0$. With $h = 0$ only the equilibrium position, realized only when $c = 0$, is possible. The phase portraits of the reduced system are shown in Fig.2 for the case when $\gamma > 0$ (when $\gamma < 0$ the portraits are obtained by symmetric mapping with respect to the vertical axis of the phase portraits in the case when $\gamma > 0$).

In the phase space T^*M every connected component of the set $U_{c,h}$ of levels of the first integrals H and p_2 is diffeomorphic to the two-dimensional cylinder $U_{c,h} \simeq S^1 \times R^1$ /7/.

The canonical transformation of the phase space

$$\{\pi_1, \pi_2 \bmod 2\pi, p_1, p_2\} \to \{w_1, w_2 \bmod 2\pi, I_1, I_2\}$$

(the identical transformation when $p_2 = 0$), given by

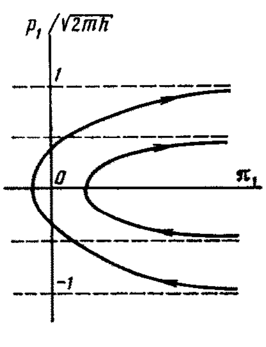

Fig.2

$$\pi_1 = -\frac{1}{2\gamma}\ln\left[\frac{I_1}{I_2} r \operatorname{sch}(\gamma w_1)\right]^2, \quad \pi_2 = w_2 + \frac{I_1}{\gamma I_2}\operatorname{th}(\gamma w_1) \tag{2.7}$$

$$p_1 = I_1 \operatorname{th}(\gamma w_1),\ p_2 = I_2\ \ (I_1 \geqslant 0)$$

reduces the Hamiltonian (2.5) and the equations of motion to the form

$$H = (2m)^{-1} I_1^2 \tag{2.8}$$

$$w_1^{\cdot} = m^{-1} I_1,\ w_2^{\cdot} = 0,\ I_1^{\cdot} = 0,\ I_2^{\cdot} = 0 \tag{2.9}$$

Note that transformation (2.7) is in essence a rectifying diffeomorphism of the Hamiltonian of a vector field with Hamiltonian (2.5).

The variables $\mathbf{I}$, $\mathbf{w}$ can be regarded as an analogue of the action-angle variables (here, the set of levels of the first integrals is not compact). The variables $w_1, w_2 \bmod 2\pi$ are the coordinates on the invariant cylinders $\mathbf{I} = \text{const}$. It follows from (2.8) that the Hamiltonian of the system is degenerate:

$$\det \| \partial^2 H/\partial \mathbf{I}^2 \| = 0$$

Eqs.(2.9) are easily integrated:

$$w_1 = \omega_1 t + w_{10}, \ w_2 = w_{20}$$
$$I_i = \text{const}, \ w_{i0} = \text{const}, \tag{2.10}$$
$$\omega_1 \equiv I_1/m \ \ (i = 1, 2)$$

To complete our study of the motion of the Chaplygin sledge over a horizontal plane, we write the expressions for transformation from the phase variables **w**, **I** to the variables $\varkappa, \ \varphi, \ p_\varkappa, \ p_\varphi$

$$\varkappa = -\frac{1}{2\gamma} \ln \left[\frac{I_1}{I_2} r \operatorname{sch}(\gamma w_1)\right]^2, \quad \varphi = w_2 - \frac{r}{\alpha} \arcsin \operatorname{sch}(\gamma w_1) \times$$
$$\operatorname{sign}[I_2 \operatorname{th}(\gamma w_1)], \quad p_\varkappa = I_1 \operatorname{th}(\gamma w_1), \quad p_\varphi = r I_1 \operatorname{sch}(\gamma w_1) \operatorname{sign} I_2 \tag{2.11}$$

Substituting (2.10) into (2.11), we find the explicit time dependences of the variables $\varkappa, \ \varphi, \ p_\varkappa, \ p_\varphi$ and their limiting values

$$\lim p_\varphi(t) = 0, \lim \varphi(t) = w_{20}, \lim p_\varkappa(t) = m\omega_1 \operatorname{sign} \alpha \ (t \to \infty) \tag{2.12}$$

Notice also that $\lim q^{\cdot}(t) = \omega_1 \operatorname{sign} \alpha \ \ (t \to \infty)$.

Consequently, in the course of time the sledge tends to uniform sliding along a straight line. The skate is then oriented in such a way that the centre of gravity G "leads" the blade A.

We now compare the behaviour of the trajectories in the phase space when the latter is covered by different maps.

It has been said that the levels of the first integrals $p_\varphi e^{\gamma\varkappa} = c$ and $H = h$ cut out in the phase space $\{\varkappa, \varphi, p_\varkappa, p_\varphi\}$ invariant manifolds $U_{c,h}$, which are diffeomorphic to one or more two-dimensional cylinders. The level $c = 0, \ h > 0$ is the union of two cylinders, i.e., $U_{0,h} = \Sigma_+ \cup \Sigma_-, \ \Sigma_+ \simeq \Sigma_- \simeq R^1 \times S^1$. The quantity $\Sigma_+ (\Sigma_-)$ is the manifold of sledge stationary motions. The sledge then moves uniformly with velocity $\sqrt{2h/m} \, (-\sqrt{2h/m})$ along the line $\varphi = \text{const}$. We fix the set of energy levels $H = h > 0$. This is the domain in the phase space between the two cylinders Σ_+ and Σ_-. It is stratified into two-dimensional phase cylinders $U_{c,h} \ (c \in R)$. Any trajectory lying in $U_{c,h} \ (c \neq 0)$ tends as $t \to +\infty \ (t \to -\infty)$ to a trajectory in $\Sigma_+ (\Sigma_-)$, if $\alpha > 0$, and to a trajectory in $\Sigma_- (\Sigma_+)$ if $\alpha < 0$. Thus the phase cylinders $U_{c,h}$ are compressed into Σ_+ as $\varkappa \to +\infty$, or into Σ_- as $\varkappa \to -\infty$.

In the phase space $\{\pi_1, \pi_2, p_1, p_2\}$ the domain $H = h > 0$ is also stratified into two-dimensional cylinders, which, however, are no longer compressed towards $\Sigma_\pm$ as $\pi_1 \to \pm\infty$ (as distinct from the previous case, where $\lim p_\varphi(t) = 0$ as $t \to \infty$; here, $p_2 = \text{const}$).

Fig.3

3. Now let the angle $\theta \neq 0$. In this case the equations of motion (1.7) have two one-parameter families of solutions $\Lambda_+ = \{\varkappa^{\cdot} = (g \sin \theta)t + v_0, \ \varphi = 0\}$ and $\Lambda_- = \{\varkappa^{\cdot} = -(g \sin \theta)t + v_0, \ \varphi = \pi\} \ (v_0 = \text{const})$, which correspond to the sledge sliding along the straight line of steepest descent with constant acceleration $g \sin \theta$. In the motion of Λ_+ the centre of gravity is ahead of the skate when $\alpha > 0$ and is behind it when $\alpha < 0$, while in the motion of Λ_- the centre of gravity is ahead of the skate when $\alpha < 0$ and is behind when $\alpha > 0$.

The remaining motions will be studied by the averaging method /8/, taking θ as a small parameter $(0 < \theta \ll 1)$.

We write the equations of motion (1.7) in the variables $I_1, \ w_1, \ w_2$

$$I_1^{\cdot} = \varepsilon \, [\operatorname{th}(\gamma w_1) \cos \varphi - \sigma \alpha r^{-1} \operatorname{sch}(\gamma w_1) \sin \varphi] \tag{3.1}$$
$$w_2^{\cdot} = -\varepsilon I_1^{-1} \, [\operatorname{th}(\gamma w_1) \sin \varphi + \sigma \alpha^{-1} r \operatorname{sch}(\gamma w_1) \cos \varphi]$$
$$w_1^{\cdot} = m^{-1} I_1 + \varepsilon \, (\gamma I_1)^{-1} \, [\cos \varphi + \sigma \alpha r^{-1} \operatorname{sh}(\gamma w_1) \sin \varphi];$$
$$\varepsilon \equiv mg \sin \theta, \ \sigma = \operatorname{sign}(I_2)$$

In (3.1) we have to replace φ by the appropriate expression of (2.11). The equation for I_2 is omitted, since it is not needed.

In system (3.1), $I_1, \ w_2$ are slow variables, and w_1 is a fast variable. On averaging the right-hand sides for the slow variables with respect to the fast variable, we obtain the averaged system of equations (we preserve the previous notation for the averaged quantities):

$$I_1^{\cdot} = \varepsilon \cos w_2, \ w_2^{\cdot} = -\varepsilon I_1^{-1} \sin w_2 \tag{3.2}$$

Notice that the terms dependent on σ disappear on averaging.

The solutions of the averaged system approximate the slow variables with an error of order θ in a time interval of order θ^{-1}.

The averaged system has the integral

$$I_1 \sin w_2 = \lambda \tag{3.3}$$

The phase portraits of system (3.2) are shown in Fig.3 ($\lambda > 0$), while analysis of the general solution of system (3.2) leads to the following limiting values:

$$\lim \sin w_2(t) = 0, \ \lim I_1^{\cdot}(t) = \pm\varepsilon \ (t \to \infty) \tag{3.4}$$

Hence it follows that the sledge will tend to descend along the line of steepest descent with constant acceleration $g \sin \theta$, while orienting itself in such a way that the centre of gravity is ahead of the blade. Thus, to a first approximation in θ, all the solutions of system (1.7) tends to Λ_+ if $\alpha > 0$ and to Λ_- if $\alpha < 0$ (only the solutions of $\Lambda_\pm$ are an exception).

Notice in conclusion that, when $\theta = 0$, $\alpha = 0$, there is a connection between the "natural" phase variables $\xi, \varphi, p_\xi, p_\varphi'$ and the variables $\varkappa, \varphi, p_\varkappa, p_\varphi$. To be more exact, transformation of the phase space

$$\varkappa = \xi - \beta \sin \varphi + (k^2 p_\xi \cos \varphi \, (\varphi - \sin \varphi))(p_\varphi' + \beta p_\xi \cos \varphi)^{-1}$$
$$p_\varkappa = p_\xi \cos \varphi, \quad p_\varphi = p_\varphi' + \beta p_\xi \cos \varphi \quad (\cos \varphi \neq 0)$$

reduces the equations of motion in the variables $\xi, \varphi, p_\xi, p_\varphi'$ to the form (1.7).

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